

# On a novel iterative method to compute polynomial approximations to Bessel functions of the first kind and its connection to the solution of fractional diffusion/diffusion-wave problems

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**Abstract.** We present an iterative method to obtain approximations to Bessel functions of the first kind  $J_p(x)$  ( $p > -1$ ) via the repeated application of an integral operator to an initial seed function  $f_0(x)$ . The class of seed functions  $f_0(x)$  leading to sets of increasingly accurate approximations  $f_n(x)$  is considerably large and includes any polynomial. When the operator is applied once to a polynomial of degree  $s$ , it yields a polynomial of degree  $s + 2$ , and so the iteration of this operator generates sets of increasingly better polynomial approximations of increasing degree. We focus on the set of polynomial approximations generated from the seed function  $f_0(x) = 1$ . This set of polynomials is not only useful for the computation of  $J_p(x)$ , but also from a physical point of view, as it describes the long-time decay modes of certain fractional diffusion and diffusion-wave problems.

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## 1. Introduction

Bessel functions play a central role in numerous problems in Science [1, 2, 3, 4, 5]. In particular, Bessel functions of the first kind  $J_p(x)$  appear in the solution of many problems involving wave and heat equations, especially in systems with spherical or cylindrical symmetry [2, 3, 4]. Such functions have been intensively studied for more than three hundred years [5] and a large body of results concerning their properties is now available [6, 7, 8]. It is well-known that they can be expressed in differential and integral forms, in terms of infinite series, as a solution of differential equations or recursive relations, etc. Given their ubiquity, their efficient numerical computation and its approximation in terms of simpler functions is an issue of great interest [9, 10, 11, 12]. In particular, the approximation of special functions in terms of polynomials has some key advantages, as the latter can be effortlessly evaluated and manipulated.

In what follows we show how to derive infinite *sets* of polynomial approximations to  $J_p$  valid for  $p > -1$  [and, of course, also for negative integer values of  $p$  by virtue of the relation  $J_{-p}(x) = (-1)^p J_p(x)$  for  $p$  integer]. Each set is generated from an initial polynomial seed function  $f_0(x)$  via the iteration of an integral operator and it eventually converges towards a properly normalized Bessel function of the first kind,  $\tilde{J}_p(x) \equiv 2^p p! J_p(z_p x) / (z_p x)^p$ , where  $z_p$  denotes the first zero of  $J_p(x)$ . When applied to a polynomial approximation of degree  $s$ , the integral operator yields an improved polynomial estimate of degree  $s + 2$ . The set of polynomial approximations  $\text{Ba}_n^{(p)}$  we primarily focus on stems from the possibly simplest seed function  $f_0(x) = 1$ . However, we also present some results for another set of polynomials  $\text{Be}_n^{(p)}$  corresponding to the choice  $f_0(x) = 1 - x$ .

Our method is inspired by the interesting properties of the integral operator

$$\Lambda_p[f] = z_p^2 \int_x^1 \frac{du}{u^{2p+1}} \int_0^u dv v^{2p+1} f(v) \quad (1)$$

recently introduced by the authors in collaboration with Borrego [13] to study Fourier-Bessel solutions of fractional diffusion equations. The operator  $\Lambda_p$  is closely related to Bessel's differential equation and it has the remarkable property of leaving the function  $\tilde{J}_p(x)$  invariant (cf. Appendix A). Interestingly enough, when the operator  $\Lambda_p$  is properly normalized, the resulting functions converge to  $\tilde{J}_p(x)$ . In the language of dynamical systems theory one could therefore respectively speak of “fixed point of the application defined by the operator” and “basin of attraction” when referring to  $\tilde{J}_p(x)$  and the set of seed functions which eventually converge to  $\tilde{J}_p(x)$ .

As an aside, it is interesting to note that our method is similar in spirit to Neumann's method for tackling integral equations, where the integral operator defining the integral equation is used to generate a function series starting from a zero-th order approximation function [15]. In contrast, our method differs significantly from the techniques used in recent works on polynomial approximations for Bessel functions. For example, Gross [10] exploited the similarity of an integral related to a problem of electromagnetic scattering in a conducting strip grating with an integral representation

of  $J_p(x)$  to devise a polynomial approximation valid for  $J_0(x)$  and  $J_1(x)$ . Millane and Eads subsequently extended this type of approximation to any  $J_p(x)$  of integer order  $p$  [11]. More recently, Li et al. generalized these results by computing approximations valid for any real  $p$  [12]. At a later stage, we shall use the results in [12] as a reference to test the accuracy of our own approximation.

The present work is organized as follows. Section 2 is devoted to the introduction of some preliminary definitions. In section 3, the operator  $\Lambda_p$  is shown to have an attractor proportional to  $\tilde{J}_p(x)$ . We demonstrate that repeatedly applying  $\Lambda_p$  to any non-zero polynomial seed function generates a set of polynomials which converge to the attractor. In section 4 we show that the attractor of a slightly modified operator  $\hat{\Lambda}_p$  is  $\tilde{J}_p(x)$  itself. We subsequently study the set of polynomial approximations generated by two different seed functions and discuss some numerical results (section 5). As shown in section 6, one of these sets appears naturally in the context of some fractional diffusion/diffusion-wave problems, thereby describing the long-time decay of the solutions. Finally, we summarize our main conclusions and briefly outline some avenues for further research in section 7.

## 2. Preliminary definitions

For the purpose of finding polynomial approximations to  $J_p(x)$ , it is convenient to introduce the following set of normalized functions

$$\tilde{J}_{p,n}(x) = 2^p p! (z_{p,n} x)^{-p} J_p(z_{p,n} x), \quad (2)$$

where  $z_{p,n}$  is the  $n$ -th zero of  $J_p(x)$ . The function  $\tilde{J}_{p,1}(x)$  will hereafter play a central role, and hence we shall use the short-hand notation  $\tilde{J}_p(x)$  to denote it. Likewise we shall use  $z_{p,1} \equiv z_p$ . In terms of the above notation, (2) can be rewritten as

$$J_p(x) = \frac{x^p}{2^p p!} \tilde{J}_p(x/z_p) \quad (3)$$

for  $n = 1$ . The last equation implies that, if polynomial estimates for  $\tilde{J}_p(x)$  are available, they immediately lead to approximations of the same type for  $J_p(x)$ . From the series representation of  $J_p(x)$ ,

$$J_p(x) = \frac{x^p}{2^p} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+p)!} \left(\frac{x}{2}\right)^{2k}, \quad (4)$$

one easily sees that  $\tilde{J}_p(0) = 1$ .

Let us introduce the following linear integral operator

$$\Lambda_{p,n}[f] = z_{p,n}^2 \int_x^1 \frac{du}{u^{2p+1}} \int_0^u dv v^{2p+1} f(v) \quad (5)$$

acting on the set of functions  $\mathcal{C}$  whose elements are those functions for which the double integral performed by  $\Lambda_{p,n}$  is well defined. It is easy to show that any function which is equally singular or less singular than  $x^\alpha$  with  $\alpha > \max[-2, -2-2p]$  is an element of  $\mathcal{C}$ . [We say that  $g(x)$  and  $x^\alpha$  are equally singular if  $g(x)/x^\alpha \rightarrow \text{const.}$  as  $x \rightarrow 0$ ;  $g(x)$  is less

singular than  $x^\alpha$  if  $g(x)/x^\alpha \rightarrow 0$  as  $x \rightarrow 0$ ]. Note also that any polynomial function is mapped onto another polynomial function by the operator  $\Lambda_{p,n}$ .

As shown in section Appendix A, the functions  $\tilde{J}_{p,n}(x)$  remain invariant under the action of the operator  $\Lambda_{p,n}$ , i.e.,

$$\Lambda_{p,n} [\tilde{J}_{p,n}(x)] = \tilde{J}_{p,n}(x). \quad (6)$$

In what follows we shall use the short-hand notation  $\Lambda_p \equiv \Lambda_{p,1}$  for simplicity. With this notation, one has

$$\Lambda_p [\tilde{J}_p(x)] = \tilde{J}_p(x) \quad (7)$$

for the special case  $n = 1$ .

### 3. The operator $\Lambda_p$ has an attractor proportional to $\tilde{J}_p(x)$

We aim to show that

$$\lim_{n \rightarrow \infty} \Lambda_p^n [P_s(x)] \propto \tilde{J}_p(x), \quad (8)$$

where  $P_s(x) = \sum_{r=0}^s c_r x^r$  is a polynomial of degree  $s$ . To accomplish this task, we shall first show that  $\Lambda_p^n [1]$  has an attractor proportional to  $\tilde{J}_p(x)$ . Next, we shall derive an expression for  $\Lambda_p^n [P_s(x)]$  in terms of  $\Lambda_p^0 [1], \dots, \Lambda_p^n [1]$ . Finally, we shall argue that the limit  $n \rightarrow \infty$  of that expression yields (8).

For the sake of simplicity, we shall use the notation  $\Lambda_p^m [1] = I_m(x)$  in what follows (in this short-hand notation, no explicit reference to the index  $p$  is made, which should be inferred from the context). Clearly, the  $I_m(x)$ 's are polynomial functions of degree  $2m$  in  $x$ . In particular, one of course has  $\Lambda_p^0 [1] = I_0(x) = 1$ . In order to study the behaviour of these functions as  $m \rightarrow \infty$ , we now invoke a relation obtained from (B4), (9) and (11) in reference [13], namely

$$I_m(x) = \Lambda_p^m [1] = \frac{1}{2^{p-1} p!} \sum_{k=1}^{\infty} \left( \frac{z_p}{z_{p,k}} \right)^{2m} \frac{z_{p,k}^{p-1}}{J_{p+1}(z_{p,k})} \tilde{J}_{p,k}(x) \quad (9)$$

(for a short demonstration, see Appendix B). Note that, as  $m$  gets larger, the weight of the large- $k$  terms in the above sum decreases very rapidly due to the prefactor  $z_{p,k}^{-2m}$ . In the limit  $m \rightarrow \infty$ , the only relevant contribution corresponds to the  $k = 1$  term. Hence, one has

$$I_m(x) \rightarrow \zeta_p \tilde{J}_p(x), \quad \text{as } m \rightarrow \infty, \quad (10)$$

where the number  $\zeta_p$  is

$$\zeta_p = \frac{1}{2^{p-1} p!} \frac{z_p^{p-1}}{J_{p+1}(z_p)}. \quad (11)$$

Because of the linearity of  $\Lambda_p^m$ , formula (10) can be rewritten as

$$\lim_{m \rightarrow \infty} \Lambda_p^m [1/\zeta_p] = \tilde{J}_p(x). \quad (12)$$

Equations (12) and (7) provide two remarkable ways of defining  $p$ -th order Bessel functions of the first kind by means of the integral operator  $\Lambda_p$ : (i) as the limit of the

feedback process  $f_{n+1}(x) = \Lambda_p[f_n(x)]$  with  $f_0(x) = 1/\zeta_p$ , and (ii) as the solution of the fixed point equation  $f(x) = \Lambda_p[f(x)]$ . These definitions nicely resemble the way in which fractals are defined in terms of the Hutchinson operator [14]. Of course, equation (10) also allows one to compute increasingly accurate estimates for the normalized functions  $\tilde{J}_p(x)$  and thus for  $J_p(x)$  by virtue of (3).

### 3.1. Evaluation of $\Lambda_p^n[P_s(x)]$

A straightforward application of the operator  $\Lambda_p^n$  to the monomial  $x^r$  gives

$$\Lambda_p[x^r] = \frac{1 - x^{r+2}}{(2+r)(2+2p+r)} = a_r - a_r x^{r+2} \quad \text{for } p > -1, r \geq 0, \quad (13)$$

where  $a_r = z_p^2/[(2+r)(2+2p+r)]$ . It can be proven by induction that

$$\Lambda_p^n[x^r] = \sum_{k=1}^n (-1)^{k-1} b_k(r, p) I_{n-k}(x) + (-1)^n b_n(r, p) x^{r+2n}. \quad (14)$$

In the above expression we have introduced

$$b_k(r, p) \equiv \prod_{m=0}^{k-1} a_{r+2m} = z_p^{2k} \frac{r!!}{(r+2k)!!} \frac{(2p+r)!!}{(r+2p+2k)!!} \quad (15)$$

as well as the definitions  $(2n+1)!! = (2n+1)!/(2^n n!)$  and  $(2n)!! = 2^n n!$ . Note that  $b_n(r, p)$  goes rapidly to zero for large  $n$ :

$$b_n(r, p) \sim (z_p/2)^{2n}/(n!)^2. \quad (16)$$

Applying  $\Lambda_p^n$  to an  $s$  degree polynomial  $P_s(x) = \sum_{r=0}^s c_r x^r$  and using (15) and (14), one gets

$$\Lambda_p^n[P_s(x)] = \sum_{k=1}^n (-1)^{k-1} I_{n-k}(x) \sum_{r=0}^s c_r b_k(r, p) + (-1)^n x^{2n} \sum_{r=0}^s c_r b_n(r, p) x^r. \quad (17)$$

Next, let us investigate the behaviour of the above expression when  $n \rightarrow \infty$ . First note that, according to (16), the coefficients  $b_n(r, p)$  roughly go to zero as  $(n!)^{-2}$  for large  $n$ . As a result of this, the second term on the r.h.s. of (17) becomes negligible with respect to the first one, which is a polynomial of degree  $2n-2$  with an independent term (of course, the larger the value of  $x$ , the larger the value of  $n$  necessary to make this term negligible). On the other hand, it is not difficult to see that the first term of (17) tends to an expression proportional to  $\tilde{J}_p(x)$ , since (i) according to (10), when  $n-k \rightarrow \infty$  the polynomials  $I_{n-k}(x)$  tend to the limiting expression  $\zeta_p \tilde{J}_p$ , and (ii) those terms in the sum for which  $n-k$  is small, i.e., those terms for which  $k$  is large, become negligible due to the fast decay of  $b_k(r, p)$  with increasing  $k$ . We thus conclude that  $\Lambda_p^n[P_s(x)] \propto \tilde{J}_p(x)$  when  $n \rightarrow \infty$ .

#### 4. Generation of $\tilde{J}_p(x)$ via the normalized operator $\hat{\Lambda}_p$

As we have just seen, when successively applied to a polynomial, the operator  $\Lambda_p$  generates another polynomial which rapidly approaches  $\tilde{J}_p(x)$  up to a prefactor. A possible way to get rid of the prefactor is to divide these polynomials by their value at the origin. The resulting “normalized” polynomials becomes equal to one at  $x = 0$ , which is precisely the value taken by  $\tilde{J}_p(x)$  at  $x = 0$ . This prompts us to introduce the “normalized” operator

$$\hat{\Lambda}_p[f] = \frac{\Lambda_p[f]}{\Lambda_p[f]_{x=0}} \quad (18)$$

acting on the subset of functions  $f \in \mathcal{C}$  for which  $\Lambda_p[f]_{x=0}$  (that is,  $\Lambda_p[f]$  evaluated at  $x = 0$ ) is non-zero. In particular, using (17), we have

$$\hat{\Lambda}_p^n[P_s(x)] = \frac{\sum_{k=1}^n (-1)^{k-1} I_{n-k}(x) \sum_{r=0}^s c_r b_k(r, p) + (-1)^n x^{2n} \sum_{r=0}^s c_r b_n(r, p) x^r}{\sum_{k=1}^n (-1)^{k-1} I_{n-k}(0) \sum_{r=0}^s c_r b_k(r, p)} \quad (19)$$

and  $\hat{\Lambda}_p^n[P_s(x)] \rightarrow \tilde{J}_p(x)$  as  $n \rightarrow \infty$ . In view of the above derivation, one expects that any well-behaved  $f(x)$  (in the sense that it can be approximated arbitrarily well by a polynomial) also converges to the same attractor, i.e.,

$$\hat{\Lambda}_p^n[f(x)] \rightarrow \tilde{J}_p(x), \quad n \rightarrow \infty. \quad (20)$$

The iteration of the operator  $\hat{\Lambda}_p$  will allow us to generate families of approximations to  $J_p(x)$  by using different seed functions  $f_0(x)$ . Each of the seed functions  $f_0(x)$  leads to a series of functions  $\{f_0, f_1, f_2, \dots\}$ , with  $f_n = \hat{\Lambda}_p^n[f_0]$ , that converges to  $\tilde{J}_p(x)$  as  $n \rightarrow \infty$ . One expects that the convergence to the attractor becomes faster as the initial condition gets closer to  $\tilde{J}_p(x)$ . On the other hand, from a practical point of view it would be desirable that the chosen seed function  $f_0$  is simple enough to ensure that the integrals resulting from the iteration of  $\hat{\Lambda}_p$  are known and easy to calculate analytically. Obvious candidates are polynomials. Surely, the most simple initial function is  $f_0(x) = 1$ . This function gives rise to the set  $\{f_n(x) \equiv \text{Ba}_n^{(p)}(x)\}$ , defined as

$$\text{Ba}_n^{(p)}(x) \equiv \hat{\Lambda}_p^{(n)}[1] = \frac{I_n(x)}{I_n(0)}. \quad (21)$$

The initial condition  $f_0(x) = P_s(x) = 1$  is a particular case of a polynomial function for which  $c_0 = 1$  and  $c_r = 0$  ( $r > 0$ ). Inserting this expression into (17) and recalling the definition of  $I_n(x)$  one gets

$$I_n(x) = \sum_{k=1}^n (-1)^{k-1} b_k(0, p) I_{n-k}(x) + (-1)^n b_n(0, p) x^{2n}, \quad (22)$$

where  $b_k(0, p) = z_p^{2k} p! / [2^{2k} k! (p+k)!]$  [cf. (15)]. Inserting this expression into (22) one finds

$$I_n(x) = \sum_{k=1}^n (-1)^{k-1} \frac{z_p^{2k} p!}{2^{2k} k! (p+k)!} I_{n-k}(x) + (-1)^n \frac{z_p^{2n} p!}{2^{2n} n! (p+n)!} x^{2n}. \quad (23)$$

The first few polynomials  $\text{Ba}_n^{(p)}(x)$  computed from (21) and (23) read

$$\begin{aligned}\text{Ba}_0^{(p)}(x) &= 1, \\ \text{Ba}_1^{(p)}(x) &= 1 - x^2, \\ \text{Ba}_2^{(p)}(x) &= 1 - \frac{2(p+2)}{p+3}x^2 + \frac{p+1}{p+3}x^4, \\ \text{Ba}_3^{(p)}(x) &= 1 - \frac{3(p+3)^2}{p^2+8p+19}x^2 + \frac{3(p+1)(p+3)}{p^2+8p+19}x^4 - \frac{(p+1)^2}{p^2+8p+19}x^6.\end{aligned}$$

As a test of the present method, we shall also investigate the behaviour of another set of functions  $\text{Be}_n^{(p)}(x)$  generated from the  $f_0(x) = 1 - x$ . In this case, from (17) and the definition of  $\hat{\Lambda}_p^n$  one finds

$$\hat{\Lambda}_p^n[1-x] = \frac{\sum_{k=1}^n (-1)^{k-1} I_{n-k}(x) [b_k(0, p) - b_k(1, p)] + (-1)^n x^{2n} [b_k(0, p) - b_k(1, p)x]}{\sum_{k=1}^n (-1)^{k-1} I_{n-k}(0) [b_k(0, p) - b_k(1, p)]}. \quad (24)$$

The corresponding polynomials can be easily evaluated via the equations (23) and (15). The first few polynomials are given below:

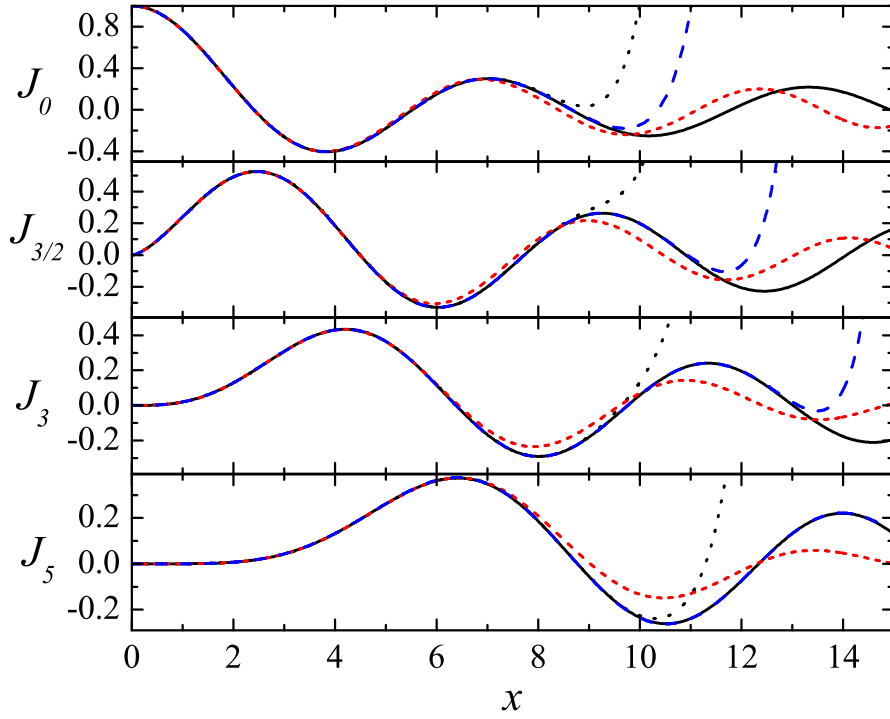
$$\begin{aligned}\text{Be}_0^{(p)}(x) &= 1 - x, \\ \text{Be}_1^{(p)}(x) &= 1 - \frac{6p+9}{2p+5}x^2 + \frac{4(p+1)}{2p+5}x^3, \\ \text{Be}_2^{(p)}(x) &= 1 - \frac{10(p+2)(2p+5)^2}{3(4p^3+36p^2+115p+113)}x^2 + \frac{5(p+1)(4p^2+16p+15)}{4p^3+36p^2+115p+113}x^4 \\ &\quad - \frac{32(p+1)^2(p+2)}{3(4p^3+36p^2+115p+113)}x^5.\end{aligned}$$

## 5. Numerical results

Li, Li and Gross (LLG) proposed in Ref. [12] an approximation to  $J_p(x)$  based on its integral representation. They obtained an infinite series whose truncation leads to the following polynomial approximation of  $J_p(x)$ ,

$$L_n^{(p)} = \sum_{m=0}^n (-1)^m \frac{n^{1-2m} (m+n-1)!}{m!(n-m)!\Gamma(m+p+1)} \left(\frac{x}{2}\right)^{2m+p} \quad (25)$$

thereby providing an alternative approximation to truncated Taylor series. Figure 1 in [12] depicts a comparison of  $J_p(x)$  with the LLG polynomial approximation of degree  $2 \times 10 + p$  and with the Taylor series truncated to the same degree for  $p = 0, 3/2, 3, 5$ . In figure 1 of the present paper we have superposed their results with our own results based on the approximation obtained by making the replacement  $\tilde{J}_p(x/z_p) \rightarrow \text{Ba}_{10}^{(p)}(x/z_p)$  in Eq. (3). We see that our approximation performs very well over a larger range of  $x$  values than the LLG polynomial approximation and the truncated Taylor series of the same order. Even though the LLG polynomial approximation oscillates around  $J_p(x)$  over a larger  $x$  interval than the other two approximations, it starts deviating from the exact curve at smaller values of  $x$ . In all cases, deviations from  $J_p(x)$  are shifted to larger  $x$  values with increasing order  $p$ . In this sense, all three approximations become better



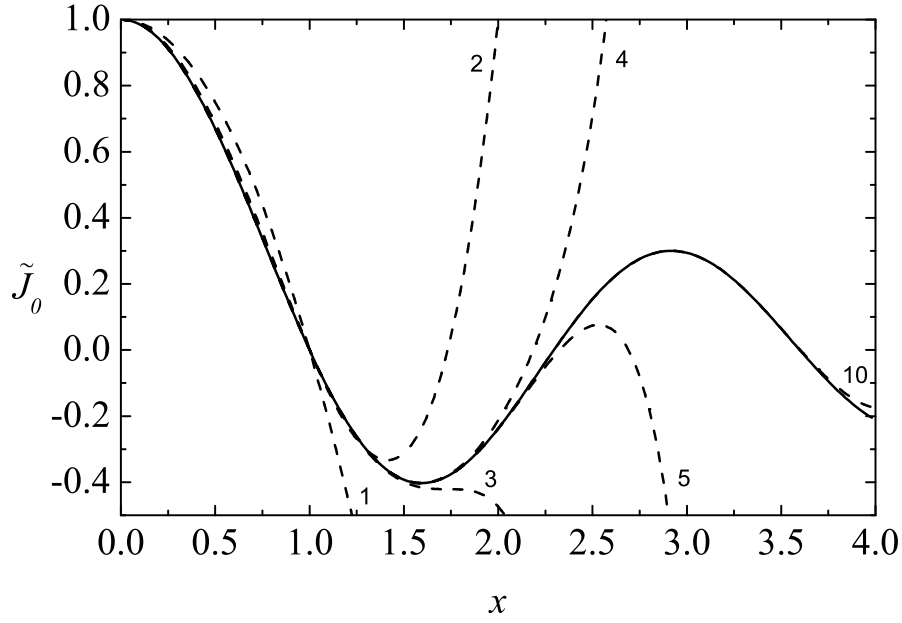
**Figure 1.** Comparison of  $J_p(x)$  (solid line) with the LLG polynomial approximation of order  $2 \times 10 + p$  [12],  $L_{10}^{(p)}$ , (short dashed line), the truncated Taylor series (dotted line) and the polynomial approximation  $Ba_p^{(10)}(x)$  of  $\tilde{J}_p(x)$  (dashed line) [see (21) and (23)] for  $p = 0, 3/2, 3, 5$ . In the lower panel the line corresponding to our approximation for  $J_5(x)$  lies on top of the exact one.

with increasing  $p$ . Regarding the computational efficiency, the CPU time employed for calculating the polynomial approximations obviously depends on their order  $n$ . For example, when using the program *Mathematica*, the CPU time necessary for computing the curves in the top panel of figure 1 (corresponding to  $n = 10$ ) is roughly the same for each of the three polynomial approximations depicted; this time is roughly one half of the time required to compute the curves corresponding to  $n = 20$  and, also, one half of the time required when using directly the Bessel function algorithm.

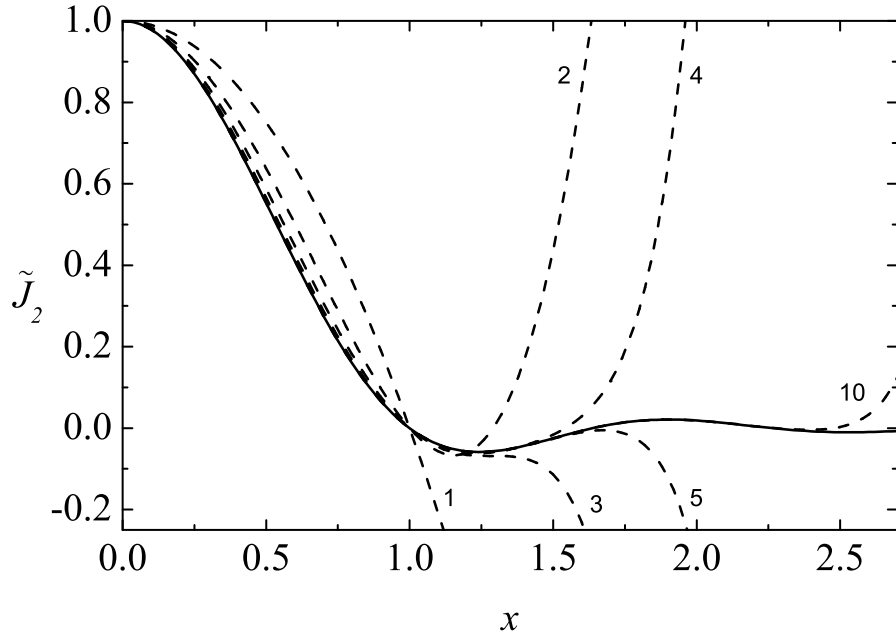
In figure 2 [figure 3] we compare  $\tilde{J}_0(x)$  [ $\tilde{J}_2(x)$ ] with the polynomials  $Ba_n^{(0)}(x)$  [ $Ba_n^{(2)}(x)$ ] for  $n = 1, 2, 3, 4, 5, 10$ . For fixed values of  $n$  and  $x$  the approximation is more accurate for  $p = 0$  than for  $p = 2$ . For other  $p$  values we have checked that, in general, the accuracy of our approximation for  $\tilde{J}_p(x)$  decreases with increasing  $p$ , as opposed to the behaviour observed for  $J_p(x)$  (see figure 1). Clearly, the different behaviour is related to the rescaling introduced by (3).

Finally, we proceed to compare  $\tilde{J}_1(x)$  with the polynomial approximations  $Ba_n^{(1)}(x)$  and  $Be_n^{(1)}(x)$  for  $n = 1, 2, 3, 4, 5, 10$ . This comparison is shown in figure 4. Not surprisingly, the family  $Be_n^{(1)}(x)$  performs better than  $Ba_n^{(1)}(x)$  since it starts from the seed function  $f_0(x) = 1 - x$  which is a better approximation to  $\tilde{J}_1(x)$  than  $f_0(x) = 1$ .

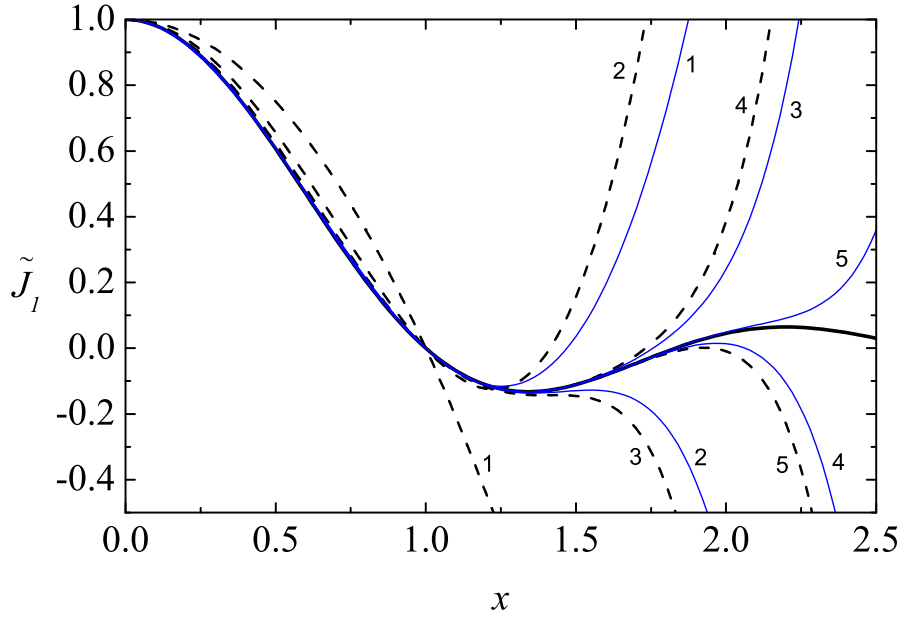




**Figure 2.** Comparison between  $\tilde{J}_0(x)$  (solid line) and  $Ba_n^{(0)}(x)$  with  $n = 1, 2, 3, 4, 5, 10$ . The value of  $n$  corresponding to each line is shown.



**Figure 3.** Comparison between  $\tilde{J}_2(x)$  (solid line) and  $Ba_n^{(2)}(x)$  with  $n = 1, 2, 3, 4, 5, 10$ . The value of  $n$  corresponding to each line is shown.



**Figure 4.** Comparison between  $\tilde{J}_1(x)$  (thick solid line) and  $Ba_n^{(1)}(x)$  (dashed lines) and  $Be_n^{(0)}(x)$  (thin solid lines) for  $n = 1, 2, 3, 4, 5$ . The value of  $n$  corresponding to each line is shown.

## 6. The polynomials $Ba_n^{(p)}(x)$ describe fractional diffusive and oscillatory modes

As shown in [13], the polynomials  $Ba_n^{(p)}(x)$  can be used to express the long-time decay modes in some fractional diffusion problems in a  $d$ -dimensional sphere. In what follows we shall demonstrate that the  $Ba_n^{(p)}(x)$ 's also appear in problems whose solution is given by the fractional diffusion-wave equation subject to the same geometry and boundary conditions (note, however, that for diffusion-wave problems one must additionally specify the initial velocity). Take, for instance, the problem described by the equation [16]

$$\frac{d^\gamma c(r, t)}{dt^\gamma} = K_\gamma \nabla^2 c(r, t), \quad (26)$$

where  $K_\gamma$  is a coefficient and the operator  $d^\gamma/dt^\gamma$  denotes the Caputo fractional derivative of order  $\gamma$  [16, 17, 18]:

$$\frac{d^\gamma f(t)}{dt^\gamma} = \frac{1}{\Gamma(n - \gamma)} \int_0^t \frac{f^n(\tau)}{(t - \tau)^{\gamma+1-n}} d\tau, \quad n - 1 < \gamma < n \quad (27)$$

with  $n$  an integer. In [13], problem B in Section III, the long-time solution of (26) corresponding to the boundary condition  $c(R, t) = 0$  and the initial condition  $c(r, 0) = c_0$  was discussed for the case  $0 < \gamma \leq 1$ . Physically,  $c(r, t)$  represents the decay of a homogeneous initial particle concentration inside a hyperspherical volume with an absorbing boundary of radius  $R$ . The solution consists of a series of decay modes whose spatial part is expressible in terms of the polynomials  $Ba_n^{(p)}(x)$  via a recursion relation essentially identical with (23).

In the range of values  $1 < \gamma < 2$ , (26) becomes a diffusion-wave equation [16, 19, 20, 21]. For  $c(r, 0) = c_0$  and  $dc(r, t)/dt = 0$  at  $t = 0$  the solution reads

$$c(r, t)/c_0 = 2(r/R)^{-\eta} \sum_{j=1}^{\infty} \frac{J_{\eta}(z_{\eta,j} r/R)}{z_{\eta,j} J_{\eta+1}(z_{\eta,j})} E_{\gamma} \left[ -(z_{\eta,j}/R)^2 K_{\gamma} t^{\gamma} \right] \quad (28)$$

with  $\eta = d/2 - 1$ , which is also the solution for  $0 < \gamma \leq 1$  (fractional diffusion equation) when  $c(r, 0) = 0$ . The solution is obtained by separation of variables and subsequent use of the relation [16, 17]

$$\frac{d^{\gamma}}{dt^{\gamma}} E_{\gamma} [-\omega^{\gamma} t^{\gamma}] = -\omega^{\gamma} E_{\gamma} [-\omega^{\gamma} t^{\gamma}] \quad (29)$$

for the derivative of the Mittag-Leffler function  $E_{\gamma}[\cdot]$ . Inserting the asymptotic expansion of the Mittag-Leffler functions

$$E_{\gamma} [-z] = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{\Gamma(1 - m\gamma)} z^{-m}, \quad z \rightarrow \infty \quad (30)$$

for  $0 < \gamma < 1$  and  $1 < \gamma < 2$  into (28) and grouping the terms with the same power of  $t$ , one finds

$$c(r, t)/c_0 = \sum_{m=1}^{\infty} \frac{(-1)^{m+1} I_m(0)}{z_p^{2m} \Gamma(1 - m\gamma)} \left( \frac{R^2}{K_{\gamma} t^{\gamma}} \right)^m \text{Ba}_m^{(\eta)}(r/R) \quad (31)$$

for  $0 < \gamma < 1$  and  $1 < \gamma < 2$  (in the above equation  $I_m(0)$  must be evaluated for  $p = \eta$ ). Thus, the spatial dependence of the long-time solution can be expressed in terms of a suitable superposition of the “fractional modes”  $\text{Ba}_m^{(\eta)}(x)$ . The above fractional solution extends the classical solution (cases  $\gamma = 1, 2$ ), where the spatial modes are proportional to Bessel functions. In other words, the role of the  $m$ -th  $d$ -dimensional normal mode  $J_{\eta}(z_{\eta,m} r/R)/(r/R)^{\eta}$  in the normal diffusion equation ( $\gamma = 1$ ) and wave equation ( $\gamma = 2$ ) is played by the  $m$ -th  $d$ -dimensional fractional mode  $\text{Ba}_m^{(\eta)}(r/R)$  in the fractional diffusion equation ( $0 < \gamma < 1$ ) and fractional diffusion-wave equation ( $1 < \gamma < 2$ ):

$$\frac{J_{\eta}(z_{\eta,m} r/R)}{(r/R)^{\eta}} \longleftrightarrow \text{Ba}_m^{(\eta)}(r/R). \quad (32)$$

## 7. Summary and Outlook

The present work deals with a novel method to obtain polynomial approximations to Bessel functions of the first kind. In some cases the obtained polynomials turn out to be more accurate than truncated Taylor series and the polynomials in [12] over a wide range of parameter values. We have seen that the polynomials  $\text{Ba}_n^{(p)}(x)$  generated by the initial function  $f_0(x) = 1$  are interesting in their own right, as they represent the spatial modes describing the long-time behaviour of certain solutions of fractional diffusion and diffusion-wave equations.

From a computational point of view, a nice property of our method is that for fixed  $x$  and  $p$  the distance  $|\text{Ba}_{n+1}^{(p)}(x) - \text{Ba}_n^{(p)}(x)|$  between successive approximations decreases rapidly with increasing iteration number  $n$ , and one can set a threshold value below

which, for practical purposes, convergence to the corresponding Bessel function may be considered to have taken place.

Integral operator methods developed along similar lines might be able to provide alternative polynomial approximations in the range  $p \leq -1$ . Of course our solution is also valid for negative integer values of  $p$  by virtue of the relation  $J_{-p}(x) = (-1)^p J_p(x)$  for  $p$  integer. Such techniques might also be useful for other kinds of Bessel functions (e.g. Bessel functions of the second and the third kind). An open question is whether similar iterative methods can be applied to generate polynomial approximations to other special functions associated with ordinary differential equations.

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## Appendix A. Invariance of $\tilde{J}_{p,n}(x)$ under the action of $\Lambda_{p,n}$

The starting point is the differential equation fulfilled by the Bessel functions

$$y^2 \frac{d^2 J_p(y)}{dy^2} + y \frac{dJ_p(y)}{dy} + (y^2 - p^2) J_p(y) = 0. \quad (\text{A.1})$$

Using the transformation  $y = z_{p,n} v$  in the above equation gives

$$v^2 \frac{d^2 J_p(z_{p,n} v)}{dv^2} + v \frac{dJ_p(z_{p,n} v)}{dv} + (z_{p,n}^2 v^2 - p^2) J_p(z_{p,n} v) = 0. \quad (\text{A.2})$$

Multiplying by  $v^{p-1}/z_{p,n}^2$  and rearranging terms we get

$$-v^{p+1} J_p(z_{p,n} v) = \frac{v^{p+1}}{z_{p,n}^2} \frac{d^2 J_p(z_{p,n} v)}{dv^2} + \frac{v^p}{z_{p,n}^2} \frac{dJ_p(z_{p,n} v)}{dv} - \frac{p^2 v^{p-1}}{z_{p,n}^2} J_p(z_{p,n} v). \quad (\text{A.3})$$

The above equation can be rewritten as follows

$$-v^{p+1} J_p(z_{p,n} v) = \frac{1}{z_{p,n}^2} \frac{d}{dv} v^{2p+1} \frac{d}{dv} [v^{-p} J_p(z_{p,n} v)]. \quad (\text{A.4})$$

Integrating between 0 and  $u$  we get

$$-\frac{z_{p,n}^2}{u^{2p+1}} \int_0^u dv v^{p+1} J_p(z_{p,n} v) = \frac{d}{du} [u^{-p} J_p(z_{p,n} u)], \quad (\text{A.5})$$

where we have used

$$\begin{aligned} \lim_{v \rightarrow 0} v^{2p+1} \frac{d}{dv} (v^{-p} J_p(z_{p,n} v)) &= \lim_{v \rightarrow 0} \left[ v^{p+1} \frac{dJ_p(z_{p,n} v)}{dv} - p v^p J_p(z_{p,n} v) \right] \\ &= -z_{p,n} \lim_{v \rightarrow 0} v^{p+1} J_{p+1}(z_{p,n} v) = 0 \quad (p > -1). \end{aligned} \quad (\text{A.6})$$

Integrating once again (A.5) between 1 and  $x$  and using  $J_p(z_{p,n}) = 0$  we get

$$z_{p,n}^2 \int_x^1 \frac{du}{u^{2p+1}} \int_0^u dv v^{2p+1} v^{-p} J_p(z_{p,n} v) = x^{-p} J_p(z_{p,n} x). \quad (\text{A.7})$$

Finally, multiplying the above equation by  $2^p p! z_{p,n}^{-p}$  yields (6).

## Appendix B. Proof of formula (9)

The Fourier-Bessel expansion of a function  $g(x)$  is given by  $g(x) = \sum_{k=1}^{\infty} c_k J_p(z_{p,k} x)$  with

$$c_k = \frac{2}{J_{p+1}^2(z_{p,k})} \int_0^1 x g(x) J_p(z_{p,k} x) dx. \quad (\text{B.1})$$

Therefore, (9) is equivalent to the statement that the coefficients of the Fourier-Bessel expansion of  $x^p I_n(x)$  are

$$c_k \equiv c(k, p, n) = 2 \left( \frac{z_p}{z_{p,k}} \right)^{2n} \frac{z_{p,k}^{p-1}}{J_{p+1}(z_{p,k})}. \quad (\text{B.2})$$

We are going to prove this equation by induction. To start with, it is well-known [6] that the coefficients of the Fourier-Bessel expansion of  $x^p I_0(x) = x^p$  are

$$c(k, p, 0) = \frac{2z_{p,k}^{-1}}{J_{p+1}(z_{p,k})}. \quad (\text{B.3})$$

This justifies (B.2) for  $n = 0$ . Therefore, to prove equation (9) is equivalent to prove that  $c(k, p, n+1) = (z_p/z_{p,k})^2 c(k, p, n)$ . By using (B.1) with  $g(x) = x^p I_{n+1}(x)$  and making the substitution  $y = z_{p,k}x$  inside the integral one sees that

$$c(k, p, n+1) = \frac{2}{z_{p,k}^{p+2} J_{p+1}^2(z_{p,k})} \int_0^{z_{p,k}} y^{p+1} J_p(y) I_{n+1}(y/z_{p,k}) dy. \quad (\text{B.4})$$

Using  $y^{p+1} J_p(y) = d[y^{p+1} J_{p+1}(y)]/dy$ , and integrating by parts one gets

$$c(k, p, n+1) = -\frac{2}{z_{p,k}^{p+2} J_{p+1}^2(z_{p,k})} \int_0^{z_{p,k}} y^{p+1} J_{p+1}(y) \frac{d}{dy} I_{n+1}(y/z_{p,k}) dy \quad (\text{B.5})$$

as the boundary terms vanish. But, from (5),

$$I_{n+1}(x) = z_p^2 \int_x^1 \frac{du}{u^{2p+1}} \int_0^u dv v^{2p+1} I_n(v). \quad (\text{B.6})$$

Inserting this expression into (B.5), using the relation  $J_{p+1}(y)/y^p = -d[J_p(y)/y^p]/dy$  and integrating by parts one gets

$$c(k, p, n+1) = \frac{2z_k^2}{z_{p,k}^{p+4} J_{p+1}^2(z_{p,k})} \int_0^{z_{p,k}} y^{p+1} J_p(y) I_n(y/z_{p,k}) dy \quad (\text{B.7})$$

as the boundary terms vanish. Comparing this result with the expression of  $c(k, p, n)$  given by (B.4) one sees that  $c(k, p, n+1) = (z_k^2/z_{p,k}^2) c(k, p, n)$ , which is just the result we aimed to prove.